3236 Statistical Theory

2/16/23

Maximum likelihood and MLE.

Method of moment estimators are consistent.

What about MLE?

X_i random sample

$f(x;\theta)$

0) The p.d.f. really depend on θ

$$f(x,\theta) \neq f(x,\theta')$$

1) $f(x;\theta)$ all have the same support in x.

$$A = \{x \mid f(x;\theta) > 0\}$$

does not depend on θ.

The X_i have a given value θ_0 of the parameter.

θ_0 is the true value.

$$\lim_{N \to \infty} \mathbb{P}_{\theta_0} \left[L(\theta_0, \underline{X}) > L(\theta, \underline{X}) \right] = 1$$

$$\forall \theta \neq \theta_0$$

Remember

$$\ell(\theta, \underline{x}) = \log L(\theta, \underline{x}) =$$

$$L(\theta, \underline{x}) = \prod_{i=1}^{N} f(x_i; \theta)$$

$$L(\theta_0; \underline{x}) > L(\theta, \underline{x}) =$$

$$\Downarrow$$

$$\prod_{i=1}^{N} \frac{f(x_i; \theta)}{f(x_i, \theta_0)} < 1$$

$$\Downarrow$$

$$\frac{1}{N} \log \prod_{i=1}^{N} \frac{f(x_i; \theta)}{f(x_i; \theta_0)} < 0$$

$$\mathbb{P}_{\theta_0}\left(\frac{1}{N}\sum_{i=1}^{N}\log\frac{f(X_i;\theta)}{f(X_i;\theta_0)} < 0\right)$$

$$Y = \frac{1}{N}\sum_{i=1}^{N}\log\frac{f(X_i;\theta)}{f(X_i;\theta_0)} \xrightarrow{P}$$

$$\mathbb{E}_{\theta_0}\left(\log\frac{f(X_1;\theta)}{f(X_1;\theta_0)}\right) <$$

$$< \log \mathbb{E}_{\theta_0}\left(\frac{f(X_1;\theta)}{f(X_1;\theta_0)}\right) =$$

$$\mathbb{E}_{\theta_0}\left(\frac{f(X_1;\theta)}{f(X_1;\theta_0)}\right) = \int_{-\infty}^{\infty}\frac{f(x;\theta)}{f(x;\theta_0)}f(x;\theta_0)\,dx$$

$$= \int_{-\infty}^{\infty} f(x;\theta)\,dx = 1$$

$$Y \xrightarrow{P} a < 0$$

$$\lim_{N\to\infty}\mathbb{P}(Y<0) = 0$$

With prob 1 we have that

$$L(\theta_0 ; X) > L(\theta ; X)$$

Suppose that

$\hat{\theta}_N$ the MLE estimator

for a sample of size N

Assume that the eq

$$\partial_\theta L(\theta, x) = 0$$

has a unique solution for every

x and N.

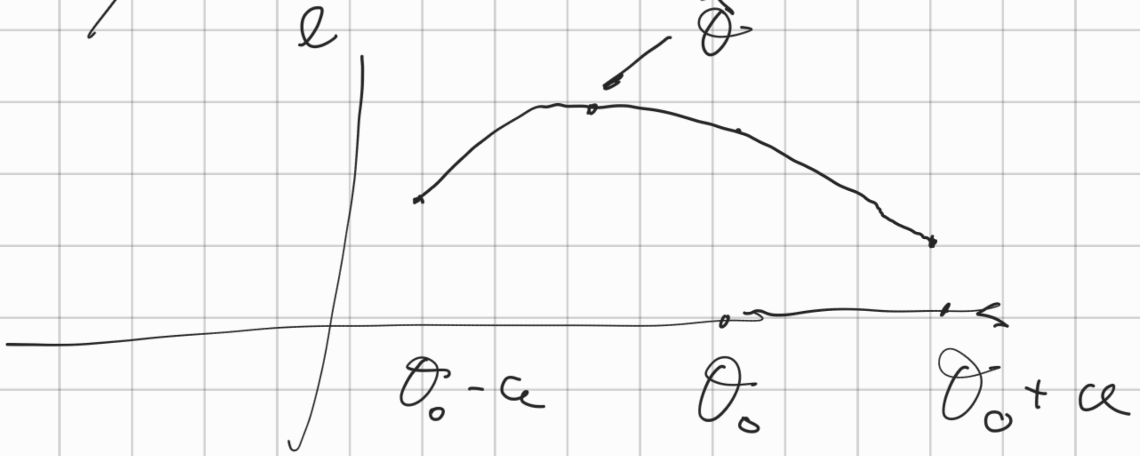
$$S_N = \{ \underline{x} \mid \ell(\theta_0, \underline{x}) > \ell(\theta_0 - a, \underline{x}) \}$$

$$\cap$$

$$\{ \underline{x} \mid \ell(\theta_0, \underline{x}) > \ell(\theta_0 + a, \underline{x}) \}$$

$$\lim_{N \to \infty} \mathbb{P}(S_N) = 1$$

For every $\underline{x}$ is S_N I have



if $\underline{x} \in S_N \implies |\hat{\theta}(\underline{x}) - \theta_0| < a$

$$\mathbb{P}(|\hat{\theta}(\underline{x}) - \theta_0| < a) > \mathbb{P}(S_N)$$

$$\lim_{N \to \infty} \mathbb{P}(|\hat{\theta}(\underline{x}) - \theta_0| < a) = 1$$

MLE estimator is consistent.

Both MME MLE are consistent

Asymptotic Distribution

X_i are r.v.

$\bar{X}$ $m(\theta)$

$\hat{\theta} = m^{-1}(\bar{X})$

$$\frac{\sqrt{N}}{\sigma}\left(\bar{X} - m(\theta_0)\right) \rightarrow N(0,1)$$

$$m'(\theta_0)\frac{\sqrt{N}}{\sigma}\left(m^{-1}(\bar{X}) - \theta_0\right) \rightarrow N(0,1)$$

The MM estimator is asympto-tically normal.

What about MLE?

$$\frac{\partial[\ln L(\hat{\theta})]}{\partial\theta} = 0$$

$$\frac{\partial[\ln L(\theta_0)]}{\partial\theta} + (\hat{\theta} - \theta_0)\frac{\partial^2[\ln L(\theta_0)]}{\partial^2\theta} = 0$$

$$\hat{\theta} - \theta_0 = \frac{\dfrac{\partial [\ln L(\theta_0)]}{\partial \theta}}{\dfrac{\partial^2 [\ln L(\theta_0)]}{\partial \theta^2}}$$

$$\frac{\partial \ln L(\theta_0)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial [\ln f(X_i ; \theta_0)]}{\partial \theta}$$

$$E_{\theta_0}\left(\frac{\partial \ln f(X_i ; \theta_0)}{\partial \theta} \right) =$$

$$\int_{-\infty}^{\infty} \frac{\partial_\theta f(x ; \theta_0)}{f(x ; \theta_0)} f(x ; \theta_0) \, dx =$$

$$\int_{-\infty}^{\infty} \partial_\theta f(x ; \theta_0) dx = \partial_\theta \left(\int_{-\infty}^{\infty} f(x ; \theta_0) dx \right)$$

$$= \partial_\theta (1) = 0$$

$$\int_{-\infty}^{\infty} \frac{\partial \ln f(x ; \theta_0)}{\partial \theta} f(x ; \theta_0) dx = 0$$

$$\int_{-\infty}^{\infty} \frac{\partial^2 \ln f(x;\theta_0)}{\partial \theta^2} f(x;\theta_0)\,dx =$$

$$- \int_{-\infty}^{\infty} \frac{\partial[\ln f(x;\theta_0)]}{\partial \theta} \frac{\partial f(x;\theta_0)}{\partial \theta}\,dx$$

$$\frac{\partial \ln f(x;\theta_0)}{\partial \theta} = \frac{\partial_\theta f(x;\theta_0)}{f(x;\theta_0)}$$

$$\frac{\partial f(x;\theta_0)}{\partial \theta} = \frac{\partial \ln f(x;\theta_0)}{\partial \theta} f(x;\theta_0)$$

$$\int_{-\infty}^{\infty} \left(\frac{\partial \ln f(x,\theta_0)}{\partial \theta}\right)^2 f(x;\theta_0)\,dx =$$

$$- \int_{-\infty}^{\infty} \frac{\partial^2 [\ln f(x;\theta_0)]}{\partial \theta^2} f(x;\theta_0)\,dx$$

$$\frac{\sqrt{n}\,(\hat{\theta} - \theta_0)}{\dfrac{1}{\sqrt{E\left[\dfrac{\partial^2 \ln f(X;\theta_0)}{\partial \theta^2}\right]}}} =$$

$$\frac{\dfrac{\partial \ln L(\theta_0)/\partial\theta}{\sqrt{n E(\partial^2 \ln f(X;\theta_0)/\partial\theta)}}}{\dfrac{-\frac{1}{n}\dfrac{\partial^2 \log L(\theta_0)}{\partial\theta}}{E\left(-\dfrac{\partial^2 \log f(X,\theta_0)}{\partial\theta^2}\right)}}$$

The r.h.s of The above

$\to N(0,1)$.

$\hat{\theta}$ MLE is asymptotically unbiased.

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \sigma)$$

$$\sigma = \frac{1}{\sqrt{-\mathbb{E}\left(\partial^2 \ln f(X; \theta_0) / \partial\theta\right)}}$$

$\hat{\theta}$ mean θ

std $\frac{1}{\sqrt{N}} \sigma$

$$\mathbb{E}\left(\partial_\theta \ln f(X, \theta)\right) = 0$$

X_i $\frac{1}{N} \sum_{i=1}^{N} \partial_\theta \ln f(X_i; \theta) = 0$